Sequential Contributions Rules for Minimum Cost Spanning Tree Problems

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Abstract

We introduce a family of sequential contributions rules for minimum cost spanning tree problems. Each member of the family assigns an agent some part of his immediate connection cost and all of his followers are equally responsible for the remaining part. We characterize the family by imposing the axioms of efficiency, non-negativity, independence of following costs, group independence, and weak first-link consistency. The Bird and the sequential equal contributions rules are two distinguished members of the sequential contributions rules. The Bird rule is obtained by requiring an agent to pay the whole part of the immediate connection cost, and the sequential equal contributions rule is obtained by requiring an agent and each of his followers to be equally responsible for the immediate connection cost. We show that how these two rules can be singled out from the family by imposing additional axioms.

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Keywords: minimum cost spanning tree problems; axiomatic characterizations; sequential contributions rules; Bird rule; sequential equal contributions rule; first-link consistency.

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1 Introduction

A minimum cost spanning tree problem is concerned with constructing a minimal cost spanning tree which provides for every node a connection to the source and allocating the construction cost among the agents. Examples of minimum cost spanning tree problems are abundant: constructing communication networks such as telephone or cable television, or building a drainage system that connects every house in a city with a water purifier. Claus and Kleitman (1973) initiated the study of the problem and Bird (1976) treated the problem with game-theoretic methods and proposed a cost allocation rule, namely the Bird rule.

After the Bird rule, many rules have been proposed for the problem: the core and the nucleolus (Granot and Huberman, 1984), the Folk solution (Feltkamp et al., 1994; Branzei et al., 2004; Bergantiños and Vidal-Puga, 2005, 2007; Lorenzo and Lorenzo-Freire, 2009), the Kar rule (Kar, 2002), the Dutta-Kar rule (Dutta and Kar, 2004), and piecewise linear solutions (Bogomolnaia and Moulin, 2008).

In this paper, we propose and characterize a new family of rules for minimum cost spanning tree problems, sequential contributions rules; each member of the family assigns an agent some part of his immediate connection cost and all of his followers are equally responsible for the remaining part.

Our main axiom is first-link consistency, which requires that upon a departure of a first-link agent, an agent directly linked to the source, the cost allocations to all of his followers should not be affected. Also we formulate its weaker version, weak first-link consistency, which requires that upon a departure of a first-link agent, the cost allocations to all of his followers should be affected by the same amount.

Our main result is a characterization of sequential contributions rules on the basis of efficiency, non-negativity, independence of following costs, group independence and weak first-link consistency. Efficiency requires that the sum of cost allocations to all agents should be equal to the cost of constructing an efficient network. Non-negativity requires that a cost allocation to each
agent should not be negative. *Independence of following costs* requires that a cost allocation of each agent should not be affected by changes in his followers’ connection costs. *Group independence* requires that for a given group, a cost allocation to a member of the group should depend only on the connection costs between themselves.\(^1\)

The Bird and the sequential equal contributions rules are two distinguished members of the sequential contributions rules.\(^2\) The Bird rule is obtained by requiring an agent to pay the whole part of his immediate connection cost, and the sequential equal contributions rule is obtained by requiring an agent and each of his followers to be equally responsible for his immediate connection cost.

Next, we show that how these two rules can be characterized by imposing additional axioms. First, the Bird rule is characterized by *efficiency*, *group independence*, and *first-link consistency*. Also, it is the unique sequential contributions rule satisfying either the requirement to be a *core selection* or *null responsibility*; a rule is a *core selection* if no coalition of agents can be better off by building their own network, and *null responsibility* requires that if an agent’s immediate connection cost is zero then the cost allocation to the agent should be zero. Finally, the Bird rule is characterized by *efficiency*, *null responsibility*, and *independence of irrelevant costs*, which requires that a cost allocation to each agent should depend only on his path to the source over a minimal cost spanning tree.

On the other hand, the sequential equal contributions rule is the unique sequential contributions rule satisfying *equal responsibility*, which requires that if a connection cost between two agents on the minimum cost spanning tree is zero, then the cost allocations to the two agents should be the same. The sequential equal contributions rule also characterized by *efficiency*, *equal*...

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\(^1\)For a problem with a unique minimum cost spanning tree, a group is defined to be a set of agents consisting of a first-link agent and all of his followers.

\(^2\)The sequential equal contributions rule is originated from the airport game, first studied by Littlechild and Owen (1973). For airport games, the sequential equal contributions rule assigns the cost of each segment equally to all airlines using the segment. The terminology we adopt is borrowed from Thomson (2005).
responsible, and independence of irrelevant costs.

The paper is organized as follows. Section 2 contains preliminaries and introduces sequential contributions rules. In section 3, we present an axiomatic characterization of sequential contributions rules. In section 4, we discuss how the Bird and the sequential equal contributions rules can be characterized by imposing additional axioms.

2 Preliminaries

2.1 The problem

Let \( N = \{1, 2, \cdots\} \) be a (finite or infinite) universe of “potential” agents and \( \mathcal{N} \) be the family of nonempty subset of \( N \). A typical element of \( \mathcal{N} \) is denoted by \( N = \{1, \cdots, n\} \). We are interested in networks whose nodes are elements of a set \( N_0 = N \cup \{0\} \), where 0 is a special node called the source. \( N_0 \equiv \{N_0 \mid N \in \mathcal{N}\} \).

Given \( N_0, t \in T_{N_0} \), and \( i \in N \), the immediate predecessor of \( i \), \( p(i; t) \), is a \( j \in N_0 \) such that \( (ij) \in t_{i_0} \) and \( p^k(i; t) \) is the \( k \)-th predecessor of \( i \), that is, \( p^k(i; t) = p^{k-1}(p(i; t); t) \). The set of all predecessors of \( i \), \( P(i; t) \), is \( \{p^k(i; t) \mid k = 1, 2, \cdots, K, \text{ where } p^K(i; t) = 0\} \). For convenience, let \( p^0(i; t) \equiv i \). The set of all followers of \( i \), \( F(i; t) \equiv \{j \in N \mid p^k(j; t) = i, k = 1, 2, 3, \cdots\} \).
When there is no ambiguity, we use $p(i)$, $P(i)$, and $F(i)$ instead of $p(i; t)$, $P(i; t)$, and $F(i; t)$. We denote the cardinality of $F(i)$ as $|F(i)|$.

For all $N_0 \in \mathcal{N}_0$ and all $C \in \mathcal{C}_{N_0}$, a minimum cost spanning tree, or an mcst, denoted by $g(C)$, is defined to be $\arg\min_{t \in T_{N_0}} \sum_{(ij) \in t} c_{ij}$, and $m(C) \equiv \min_{t \in T_{N_0}} \sum_{(ij) \in t} c_{ij}$. As established in the literature, an mcst exists even though it is not necessarily unique.

Given $N_0 \in \mathcal{N}_0$, a minimum cost spanning tree problem, or an mcstp, is a cost matrix $C \in \mathcal{C}_{N_0}$. For each $N \in \mathcal{N}$, a cost allocation rule, or a rule, is a function $\varphi : \mathcal{C}_{N_0} \to \mathbb{R}^N_+$, which associates to any problem $C \in \mathcal{C}_{N_0}$ a vector $\varphi(C) \equiv (\varphi_i(C))_{i \in N}$ in $\mathbb{R}^N_+$. For each $i \in N$, $\varphi_i(C)$ is the cost allocation to $i$.

We impose a domain restriction on the set of cost matrices:

$$C^1_{N_0} \equiv \{ C \in \mathcal{C}_{N_0} | C \text{ induces a unique mcst.} \}$$

and $\mathcal{C}^1 \equiv \bigcup_{N_0 \in \mathcal{N}_0} \mathcal{C}^1_{N_0}$. Given $C \in \mathcal{C}^1$, we denote the immediate connection cost of $i$, $c_{p(i; g(C))i}$, by $c_i$. An agent $l$ is a leaf if $F(l; g(C)) = \emptyset$ and a first-link agent if $P(l; g(C)) = \{0\}$. Let $Z(C)$ be a set of all leaves and $A(C)$ be a set of all first-link agents. For all $N_0 \in \mathcal{N}_0$, all $C \in \mathcal{C}^1_{N_0}$, and all $l \in A(C)$, $F(l; g(C)) \cup l$ is a group. Note that $N = \bigcup_{l \in A(C)} (F(l; g(C)) \cup l)$ and $(F(l; g(C)) \cup l) \cap (F(m; g(C)) \cup m) = \emptyset$ for all $l, m \in A(C)$.

### 2.2 Sequential contributions rules

We introduce a family of sequential contributions rules for minimum cost spanning tree problems. Each member of the family assigns an agent some part of his connection cost to his immediate predecessor and all of his followers are equally responsible for the remaining part. Since a leaf does not have a follower to share his immediate connection cost, we require that he is solely responsible for his connection cost.

Formally, for all $N \in \mathcal{N}$, let $\alpha \equiv (\alpha_i)_{i \in N}$ be a contributions function, where $\alpha_i$ is a function $\alpha_i : \mathcal{C}^1_{N_0} \to \mathbb{R}$ satisfying the following conditions: for all $C \in \mathcal{C}^1_{N_0}$,

1) $0 \leq \alpha_i(C) \leq c_i,$
ii) if $i \in Z(C)$, then $\alpha_i(C) = c_i$.

iii) for all $N'_0 \in N_0$ such that $i \in N'$ and all $C' \in C_{N'_0}$, if $F(i; g(C)) = F(i; g(C'))$ and $c_i = c'_i$, then $\alpha_i(C) = \alpha_i(C')$.

$\mathcal{A}_N$ be the family of all contributions functions over $N$.

We are ready to define the sequential contributions rules.

**Sequential contributions rules:** For all $N_0 \in N_0$, all $C \in C_{N_0}$, and all $\alpha \in \mathcal{A}_N$, a sequential contributions rule with a contributions function $\alpha$, $\varphi^\alpha \equiv (\varphi^\alpha_i)_{i \in N}$, is defined as

$$\varphi^\alpha_i(C) = \alpha_i(C) + \sum_{k \in P(i; g(C))} \frac{c_k - \alpha_k(C)}{|F(k; g(C))|}.$$

The Bird and the sequential equal contributions rules are two most well-known members of this family. The Bird rule is obtained by requiring each agent to be solely responsible for the immediate connection cost. The sequential equal contributions rule is obtained by requiring an agent and each of his followers to be equally responsible for the immediate connection cost. Finally, the reverse Bird rule is obtained by requiring that an agent does not pay any part of his immediate connection cost if he has a follower.

**Bird rule**, $\varphi^B$: For all $N_0 \in N_0$, all $C \in C_{N_0}$, and all $i \in N$, $\alpha_i(C) = c_i$.

**Sequential equal contributions rule**, $\varphi^E$: For all $N_0 \in N_0$, all $C \in C_{N_0}$, and all $i \in N$, $\alpha_i(C) = \frac{c_i}{|F(i)|+1}$.

**Reverse Bird rule**, $\varphi^{RB}$: For all $N_0 \in N_0$ and all $C \in C_{N_0}$, if $i \in Z(C)$, $\alpha_i(C) = c_i$ and otherwise, $\alpha_i(C) = 0$.

Even though a cost matrix does not induce a unique $mcst$, we can still define the sequential contributions rule. For each $mcst$ derived from the cost
matrix, we apply a sequential contributions rule and find the corresponding cost allocation. Each agent’s cost is calculated by taking an average of all the cost allocations.

3 The main characterization result

First, we introduce basic axioms discussed in the literature. Efficiency requires that the sum of cost allocations to all agents should be equal to the cost of constructing an efficient network. Non-negativity requires that a cost allocation to each agent should not be negative.

Efficiency: For all $N_0 \in \mathbb{N}_0$ and all $C \in C_{N_0}$, $\sum_{i \in N} \varphi_i(C) = m(C)$.

Non-negativity: For all $N_0 \in \mathbb{N}_0$, all $C \in C_{N_0}$, and all $i \in N$, $\varphi_i(C) \geq 0$.

Next are two independence axioms. Independence of following costs requires that a cost allocation of each agent should not be affected by his followers’ connection costs. Group independence requires that for a given group, a cost allocation to a member of the group should depend only on the connection costs between themselves.3 In other words, for a given group, suppose that there is a change in the connection cost between an agent in any other group and another agent – this agent can be a member of the given group. Group independence requires that as long as the connection costs between any pairs of the given group remain unchanged, such a change should not affect the cost allocation of agents in the given group.

Independence of following costs: For all $N_0 \in \mathbb{N}_0$, all $C, C' \in C_{N_0}$, and all $i \in N$, if $F(i; g(C)) = F(i; g(C'))$ and for all $q, w \in N\setminus F(i, g(C))$, $c_{qw} = c'_{qw}$, then $\varphi_i(C) = \varphi_i(C')$.

\(^3\)Kar (2002) introduces a weaker version of group independence, which requires that a change in the connection cost within a group should not affect the cost allocation of agents in any other groups.
Group independence: For all $N_0, N_0' \in \mathcal{N}_0$, all $C \in \mathcal{C}_{N_0}$, and all $C' \in \mathcal{C}_{N_0'}$, if there exists $l \in A(C) \cap A(C')$ such that $C|_{(F(l,g(C)) \cup \{l\})_0} = C'|_{(F(l,g(C')) \cup \{l\})_0}$, then for all $i \in F(l,g(C)) \cup \{l\}, \phi_i(C) = \phi_i(C')$.

Our final axiom for the main characterization is based on a certain formulation of consistency.\footnote{For a survey on consistency, see Thomson (2000).} Feltkamp et al. (1999) introduce a consistency axiom for mcstps, namely leaf consistency: it requires that a departure of a leaf should not affect the cost allocation of other remaining agents. They characterized the Bird rule on the basis of leaf consistency.

Recently, Dutta and Kar (2004) propose two other consistency axioms for mcstps. Suppose that an agent leaves the problem, but all the remaining agents are allowed to use the node. All the remaining agents can use the leaving node in two different ways:

i) source consistency, which assumes that the leaving node can be used only to connect to the source, and

ii) tree consistency, which assumes that the leaving node can be used to connect to any node.

For each formulation, consistency requires that the cost allocation of other agents should remain unchanged between the original and the reduced cost matrices. They characterize the Bird rule on the basis of source consistency and the Dutta-Kar rule on the basis of tree consistency.

In this paper, we introduce a fourth consistency axiom, first-link consistency. For all $C \in \mathcal{C}^1$ and all $l \in A(C)$, imagine that a first-link agent $l$ leaves the problem after constructing the arc $(l0)$, but remaining agents are allowed to connect to the source through the arc $(l0)$. In this reduced problem, each remaining node has an option to use the node $l$ to connect to the source. Formally, the connection costs on $N_0 \setminus \{l\}$ are changed as following:
i) for all \( i \neq l \), \( c_{i0}^{-1} = \min\{c_{i0}, c_{il}\} \),

ii) if \( \{i, j\} \cap \{l, 0\} = \emptyset \), then \( c_{ij}^{-1} = c_{ij} \).

Let \( C^{-l} \) represent the reduced cost matrix. Note that for all \( i \in \mathcal{N}\{l\} \), \( c_i^{-1} = c_i \). With a little abuse of notation, let \( C^{-P(k)} \) be the reduced cost matrix in which the agents of \( P(k)\{0\} \) leaves one by one.

**First-link consistency** requires that upon a departure of a first-link agent, all of his followers should not be affected.

**First-link consistency**: For all \( N_0 \in \mathcal{N}_0 \), all \( C \in C^1_{N_0} \), all \( l \in A(C) \), and all \( i \in F(l) \), \( \varphi_i(C) = \varphi_i(C^{-l}) \).

Next is **weak first-link consistency**, which requires that upon a departure of a first-link agent, all of his followers should be affected by the same amount. It is obvious that **first-link consistency** implies **weak first-link consistency**.

**Weak first-link consistency**: For all \( N_0 \in \mathcal{N}_0 \), all \( C \in C^1_{N_0} \), all \( l \in A(C) \), and all \( i, j \in F(l) \),

\[
\varphi_i(C) - \varphi_i(C^{-l}) = \varphi_j(C) - \varphi_j(C^{-l}).
\]

We are ready to present a characterization of the sequential contributions rules.

**Lemma 1.** On \( C^1 \), a sequential contributions rule satisfies efficiency, non-negativity, independence of following costs, group independence, and weak first-link consistency.

**Proof.** Let \( \varphi^\alpha \) be a sequential contributions rule with a contributions function \( \alpha \). It is clear that \( \varphi^\alpha \) satisfies efficiency and non-negativity.

We show that \( \varphi^\alpha \) satisfies independence of following costs. For all \( N_0 \in \mathcal{N}_0 \), all \( C, C' \in C^1_{N_0} \), and all \( i \in N \), if \( F(i; g(C)) = F(i; g(C')) = F(i) \) and for all \( q, w \in N\setminus F(i) \), \( c_{qw} = c'_{qw} \), we obtain:
i) $P(i; g(C)) = P(i; g(C'))$,

ii) for all $k \in P(i) \cup \{i\}$, $F(k; g(C)) = F(k; g(C'))$,

iii) for all $k \in P(i) \cup \{i\}$, $c_k = c'_k$,

iv) since $|F(i; g(C))| = |F(i; g(C'))|$ and $c_i = c'_i$, $\alpha_i(C) = \alpha_i(C')$.

Therefore,

$$
\varphi_i^\alpha(C) = \alpha_i(C) + \sum_{k \in P(i; g(C))} \frac{1 - \alpha_k(C)}{|F(k; g(C))|} c_k 
= \alpha_i(C') + \sum_{k \in P(i; g(C'))} \frac{1 - \alpha_k(C')}{|F(k; g(C'))|} c'_k 
= \varphi_i^\alpha(C'),
$$

the desired expression.

Next, we show that $\varphi^\alpha$ satisfies group independence. For all $N_0, N'_0 \in N_0$ such that $l \in N \cap N'$, all $C \in C^1_{N_0}$, and all $C' \in C^1_{N'_0}$, if $l \in A(C) \cap A(C')$ and $C_{[F(i; g(C))]\cup\{l\}} = C_{[F(i; g(C'))]\cup\{l\}}$, we obtain: for all $i \in F(l, g(C)) \cup \{l\}$,

i) $P(i; g(C)) = P(i; g(C'))$,

ii) for all $k \in P(i) \cup \{i\}$, $F(k; g(C)) = F(k; g(C'))$,

iii) for all $k \in P(i) \cup \{i\}$, $c_k = c'_k$,

iv) since $|F(i; g(C))| = |F(i; g(C'))|$ and $c_i = c'_i$, $\alpha_i(C) = \alpha_i(C')$.

Therefore, for all $i \in F(l, g(C)) \cup \{l\}$,

$$
\varphi_i^\alpha(C) = \alpha_i(C) + \sum_{k \in P(i; g(C))} \frac{1 - \alpha_k(C)}{|F(k; g(C))|} c_k 
= \alpha_i(C') + \sum_{k \in P(i; g(C'))} \frac{1 - \alpha_k(C')}{|F(k; g(C'))|} c'_k 
= \varphi_i^\alpha(C'),
$$

as desired.

Finally, we show that $\varphi^\alpha$ satisfies weak first-link consistency. For all $N_0 \in \mathcal{N}_0$, all $C \in C^1_{N_0}$, all $l \in A(C)$, and all $i \in F(l)$, we obtain:
i) $P(i; g(C)) = P(i; g(C^{-l})) \cup \{l\},$

ii) since $F(i; g(C)) = F(i; g(C^{-l}))$ and $c_i = c_{i^{-l}}$, $\alpha_i(C) = \alpha_i(C^{-l}).$

Therefore, for all $i \in F(l),$

$$\varphi_i^0(C) - \varphi_i^0(C^{-l}) = \left\{ \alpha_i(C) + \sum_{k \in P(i; g(C))} \frac{c_k - \alpha_k(C)}{|F(k; g(C))|} \right\}$$

$$- \left\{ \alpha_i(C^{-l}) + \sum_{k \in P(i; g(C^{-l}))} \frac{c_{k^{-l}} - \alpha_k(C^{-l})}{|F(k; g(C^{-l}))|} \right\}$$

$$= \frac{c_l - \alpha_l(C)}{|F(l; g(C))|},$$

as desired. \[\square\]

Our main characterization theorem follows.

**Theorem 1.** On $C^1$, a rule satisfies efficiency, non-negativity, independence of following costs, group independence, and weak first-link consistency if and only if it is a sequential contributions rule.

**Proof.** By Lemma 1, it suffices to show that if a rule satisfies efficiency, non-negativity, independence of following costs, group independence, and weak first-link consistency, then it is a sequential contributions rule. Let $\varphi$ be a rule satisfying the five axioms.

**Step 1:** For all $N_0 \in N_0$, all $C \in C^1_{N_0}$, and all $i \in N$, there exists $\alpha_i(C)$ such that

$$\varphi_i(C) = \alpha_i(C) + \sum_{k \in P(i; g(C))} \frac{c_k - \alpha_k(C)}{|F(k; g(C))|}. $$

**Proof.** The proof is divided into two cases.

**Case 1:** $|N| = 1$. Without loss generality, let $N \equiv \{1\}$. By efficiency, for all $C \in C^1_{N_0}$, $\varphi_1(C) = c_1$, as desired.
Case 2: $|N| \geq 2$. As induction hypothesis, suppose that for all $N'_0 \in N_0$ such that $|N'| \leq n - 1$, all $C' \in C'_{N'_0}$, and all $i' \in N'$, there exists $\alpha_i(C')$ such that

$$\varphi_i(C') = \alpha_i(C') + \sum_{k \in P(i; g(C'))} \frac{c'_k - \alpha_k(C')}{|F(k; g(C'))|}.$$  

Without loss generality, let $N = \{1, \ldots, n\}$. By weak first-link consistency, for all $C \in C_{N_0}$, all $l \in A(C)$, and all $i \in F(l; g(C))$, there exist $\gamma$ such that $\varphi_i(C) = \varphi_i(C^{-l}) + \gamma$ and by group independency, for all $i \in (N \setminus \{l\}) \setminus F(l; g(C))$, $\varphi_i(C) = \varphi_i(C^{-l})$. By efficiency,

$$\sum_{i \in N \setminus \{l\}} \varphi_i(C^{-l}) = \sum_{i \in N} \varphi_i(C) - c_l$$

$$= \sum_{i \in F(l; g(C))} \varphi_i(C) + \sum_{i \in (N \setminus \{l\}) \setminus F(l; g(C))} \varphi_i(C) - c_l$$

$$= \left\{ \sum_{i \in F(l; g(C))} \varphi_i(C^{-l}) + \left| F(l; g(C)) \right| \gamma \right\}$$

$$+ \left\{ \sum_{i \in (N \setminus \{l\}) \setminus F(l; g(C))} \varphi_i(C^{-l}) \right\} + \varphi_i(C) - c_l$$

$$= \sum_{i \in N \setminus \{l\}} \varphi_i(C^{-l}) + \left| F(l; g(C)) \right| \gamma + \varphi_i(C) - c_l,$$

which implies that $\varphi_i(C) = c_l - \left| F(l; g(C)) \right| \gamma$. Therefore, we define $\alpha_i(C) = c_l - \left| F(l; g(C)) \right| \gamma$. Then, $\varphi_i(C) = \alpha_i(C)$, the desired result.

For $i \in N \setminus \{l\}$, let $\alpha_i(C) = \alpha_i(C^{-l})$. Two subcases are possible.

Subcase 2-1: $i \in (N \setminus \{l\}) \setminus F(l; g(C))$. If $i \in (N \setminus \{l\}) \setminus F(l; g(C))$, then $\varphi_i(C) = \varphi_i(C^{-l})$, $P(i; g(C)) = P(i; g(C^{-l}))$, and for all $k \in P(i; g(C^{-l}))$,
$F(k; g(C)) = F(k; g(C^{-i})).$ Therefore, for all $i \in (N \setminus \{l\}) \setminus \{l\},$

$$\varphi_i(C) = \varphi_i(C^{-l})$$

$$= \alpha_i(C^{-l}) + \sum_{k \in P(i; g(C^{-l}))} \frac{c_k - \alpha_k(C^{-l})}{|F(k; g(C^{-l}))|}$$

$$= \alpha_i(C) + \sum_{k \in P(i; g(C))} \frac{c_k - \alpha_k(C)}{|F(k; g(C))|}.$$

**Subcase 2-2:** $i \in F(l; g(C)).$ If $i \in F(l; g(C))$, then $\varphi_i(C) = \varphi_i(C^{-l}) + \gamma,$

$P(i; g(C)) = P(i; g(C^{-l})) \cup \{l\}$, and for all $k \in P(i; g(C^{-l}))$, $F(k; g(C)) = F(k; g(C^{-l})).$ Therefore, for all $i \in F(l; g(C))$,

$$\varphi_i(C) = \varphi_i(C^{-l}) + \gamma$$

$$= \alpha_i(C^{-l}) + \sum_{k \in P(i; g(C^{-l}))} \frac{c_k - \alpha_k(C^{-l})}{|F(k; g(C^{-l}))|} + \frac{c_l - \alpha_l(C)}{|F(l; g(C))|}$$

$$= \alpha_i(C) + \sum_{k \in P(i; g(C))} \frac{c_k - \alpha_k(C)}{|F(k; g(C))|}.$$

Altogether, for all $i \in N$, there exists $\alpha_i(C)$ such that

$$\varphi_i(C) = \alpha_i(C) + \sum_{k \in P(i; g(C))} \frac{c_k - \alpha_k(C)}{|F(k; g(C))|},$$

as desired. \qed

**Step 2:** For all $N_0, N'_0 \in \mathcal{N}_0$ such that $i \in N \cap N'$, all $C \in \mathcal{C}_{N_0}^i$, and all $C' \in \mathcal{C}_{N'_0}^{i'}$ such that $F(i; g(C)) = F(i; g(C'))$ and $c_i = c_i'$,

$$\alpha_i(C) = \alpha_i(C').$$

**Proof.** For all $N_0 \in \mathcal{N}_0$, all $C \in \mathcal{C}_{N_0}^i$, and all $i \in N$, from Step 1,

$$\varphi_i(C) - \varphi_i(C^{-P(i)}) = \left\{ \alpha_i(C) + \sum_{k \in P(i; g(C))} \frac{c_k - \alpha_k(C)}{|F(k; g(C))|} \right\} - \alpha_i(C^{-P(i)}). \ (1)$$
For all \( j \in F(i) \) such that \( p(j) = i \), from Step 1,

\[
\varphi_j(C) - \varphi_j(C^{-P(i)}) = \left\{ \alpha_j(C) + \sum_{k \in P(i|g(C))} \frac{c_k - \alpha_k(C)}{|F(k; g(C))|} \right\} - \alpha_j(C^{-P(i)}) \quad (2)
\]

Since \( F(i; g(C)) = F(i; g(C^{-P(i)})) \) and \( c_i = c_i^{-P(i)} \), from (2), we have

\[
\varphi_j(C) - \varphi_j(C^{-P(i)}) = \left\{ \alpha_j(C) + \sum_{k \in P(i|g(C))} \frac{c_k - \alpha_k(C)}{|F(k; g(C))|} \right\} - \alpha_j(C^{-P(i)}). \quad (3)
\]

By weak first-link consistency, \( \varphi_i(C) - \varphi_i(C^{-P(i)}) = \varphi_j(C) - \varphi_j(C^{-P(i)}) \), or equivalently, (1) is equal to (3). Therefore,

\[
\alpha_i(C) - \alpha_i(C^{-P(i)}) = \frac{c_i - \alpha_i(C)}{|F(i; g(C))|} - \frac{c_i - \alpha_i(C^{-P(i)})}{|F(i; g(C))|} = \frac{1}{|F(i; g(C))|} \left\{ \alpha_i(C) - \alpha_i(C^{-P(i)}) \right\},
\]

or

\[
\left\{ 1 + \frac{1}{|F(i; g(C))|} \right\} \left\{ \alpha_i(C) - \alpha_i(C^{-P(i)}) \right\} = 0,
\]

which is possible if and only if

\[
\alpha_i(C) = \alpha_i(C^{-P(i)}). \quad (4)
\]

Applying the same argument to \( N_{0}^j \in \mathcal{N}_0 \) and \( C' \in \mathcal{C}_{N_{0}^j}^1 \), we have

\[
\alpha_i(C') = \alpha_i(C'^{-P(i)}). \quad (5)
\]

By group independence, \( \varphi_i(C'^{-P(i)}) = \varphi_i(C'_{|F(i)\cup\{i\} \cup\{j\}}) \) and \( \varphi_i(C'^{-P(i)}) = \varphi_i(C'_{|F(i)\cup\{i\} \cup\{j\}}) \).

Since \( i \in A(C'^{-P(i)}) \), \( \varphi_i(C'^{-P(i)}) = \alpha_i(C'^{-P(i)}) \) and \( \varphi_i(C'_{|F(i)\cup\{i\} \cup\{j\}}) = \alpha_i(C'_{|F(i)\cup\{i\} \cup\{j\}}) \).

Also, since \( i \in A(C'^{-P(i)}) \), \( \varphi_i(C'^{-P(i)}) = \alpha_i(C'^{-P(i)}) \) and \( \varphi_i(C'_{|F(i)\cup\{i\} \cup\{j\}}) = \alpha_i(C'_{|F(i)\cup\{i\} \cup\{j\}}) \). Therefore,

\[
\alpha_i(C'^{-P(i)}) = \alpha_i(C'_{|F(i)\cup\{i\} \cup\{j\}}) \quad (6)
\]
and
\[ \alpha_i(C' - P(i)) = \alpha_i(C'_{(F(i) \cup \{i\})_0}). \] (7)

By independence of following costs, \( \varphi_i(C_{[(F(i) \cup \{i\})_0]} = \varphi_i(C'_{(F(i) \cup \{i\})_0}), \) which implies that
\[ \alpha_i(C_{[(F(i) \cup \{i\})_0]} = \alpha_i(C'_{(F(i) \cup \{i\})_0}). \] (8)

Altogether,
\[ \alpha_i(C) \overset{(4)}{=} \alpha_i(C'^P(i)) \overset{(6)}{=} \alpha_i(C'_{(F(i) \cup \{i\})_0}) \overset{(8)}{=} \alpha_i(C'_{(F(i) \cup \{i\})_0}) \overset{(7)}{=} \alpha_i(C'^P(i)) \overset{(5)}{=} \alpha_i(C'), \]

the desired expression. \( \square \)

**Step 3:** For all \( N_0 \in N_0, \) all \( C \in C_{N_0}^1, \) and all \( i \in N, \)
\[ 0 \leq \alpha_i(C) \leq c_i \]
and if \( i \in Z(C) \) then \( \alpha_i(C) = c_i. \)

**Proof.** For all \( N_0 \in N_0, \) all \( C \in C_{N_0}^1, \) and all \( i \in N, \) let \( \tilde{C}^P(i) \in C_{(N \setminus P(i))_0}^1 \) be
\[ \tilde{c}_{qw}^{P(i)} = \begin{cases} 0 & \text{if } q = i \text{ and } w \in F(i), \\ \infty & \text{if } q \in (N \setminus P(i)) \setminus (F(i) \cup \{i\}) \text{ and } w \in F(i), \\ c_{qw} & \text{otherwise}. \end{cases} \]

We divide into two cases:

**Case 1:** \( i \in Z(C). \) Since \( \alpha_i(C) = \alpha_i(\tilde{C}^P(i)) \) and \( \alpha_i(\tilde{C}^P(i)) = c_i, \)
\[ \alpha_i(C) = c_i. \]

**Case 2:** \( i \in N \setminus Z(C). \) From Step 1 and 2, for all \( j \in F(i) \) such that \( p(j) = i, \)
we have
\[ \varphi_j(C^P(i)) = \alpha_j(C^P(i)) + \frac{c_i - \alpha_i(\tilde{C}^P(i))}{|F(i) \cup g(C^P(i))|}. \]
Note that $\alpha_i(C) = \alpha_i(\bar{C}^{-P(i)})$, $F(i; g(\bar{C}^{-P(i)})) = F(i; g(C))$ and $\alpha_j(\bar{C}^{-P(i)}) = 0$. Therefore,

$$\varphi_i(\bar{C}^{-P(i)}) = \frac{c_i - \alpha_i(C)}{|F(i; g(C))|}.$$  

By non-negativity, $\varphi_i(\bar{C}^{-P(i)}) \geq 0$ or equivalently $\alpha_i(C) \leq c_i$. On the other hand, since $i$ is a first-link agent of $C^{-P(i)}$, $\varphi_i(C^{-P(i)}) = \alpha_i(C)$. By non-negativity, $\varphi_i(\bar{C}^{-P(i)}) \geq 0$ or $\alpha_i(C) \geq 0$. Therefore,

$$0 \leq \alpha_i(C) \leq c_i,$$

as desired.  

By Steps 2 and 3, $\alpha$ in Step 1 satisfies the requirements to be a contributions function. Therefore, $\varphi$ is a sequential contributions rule.  

Remark 1. In this remark, we discuss what additional rules would be made possible by removing one axiom at a time from the list appearing in Theorem 1.

(i) **Efficiency**: For all $C \in \mathcal{C}_{N_0}$ and all $i \in N$, $\varphi_i^Z(C) = 0$. This zero rule assigns all agents zero. It is easy to check that this rule satisfies non-negativity, independence of following costs, group independence, and weak first-link consistency.

(ii) **Non-negativity**: For all $N_0 \in \mathcal{N}_0$ and all $i \in N$, let $\beta_i : \mathcal{C}_{N_0}^i \rightarrow \mathbb{R}$ be a function satisfying the following conditions: for all $C \in \mathcal{C}_{N_0}^i$

i) if $i \in Z(C)$, then $\beta_i(C) = c_i$,

ii) for all $N'_0 \in \mathcal{N}_0$ such that $i \in N'$ and all $C' \in \mathcal{C}_{N'_0}^i$, if $F(i; g(C)) = F(i; g(C'))$ and $c_i = c'_i$, then $\beta_i(C) = \beta_i(C')$.

Then, for all $C \in \mathcal{C}_{N_0}^i$ and all $i \in N$,

$$\varphi_i^\beta(C) = \beta_i(C) + \sum_{k \in P(i; g(C))} \frac{c_k - \beta_k(C)}{|F(k; g(C))|}$$  

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In these generalized sequential contributions rules, there is no restriction that, for all \( C \in \mathcal{C}_{N_0}^1 \) and all \( i \in N, 0 \leq \beta_i(C) \leq c_i \). It is easy to check that these rules satisfy efficiency, independence of following costs, group independence, and weak first-link consistency, but some of them may violate non-negativity.

(iii) Independence of following costs: For all \( C \in \mathcal{C}_{N_0}^1 \) and all \( i \in N \),

\[ \varphi^GE_i(C) = \frac{1}{|F(l; g(C)) \cup \{l\}|} \sum_{j \in (F(l; g(C)) \cup \{l\})} c_j, \]

where \( l \) is the unique agent such that \( l \in A(C) \cap P(i; g(C)) \). For a given group, this group-wise egalitarian rule assigns each agent in the group an equal portion of the total immediate connection costs of the group. It is easy to check that this rule satisfies efficiency, non-negativity, group independence, and weak first-link consistency.

(iv) Group independence: For all \( C \in \mathcal{C}_{N_0}^1 \) and all \( i \in N \),

\[ \varphi^1_i(C) = \begin{cases} \frac{1}{|Z(C)|} \sum_{j \in Z(C)} c_j & \text{if } i \in Z(C) \\ c_i & \text{otherwise.} \end{cases} \]

This rule assigns each leaf an equal portion of the sum of the immediate connection costs of all leaves, and any other agent his immediate connection cost as in the Bird rule. It is easy to check that \( \varphi^1 \) satisfies efficiency, non-negativity, independence of following costs, and weak first-link consistency.

(v) Weak first-link consistency: For all \( C \in \mathcal{C}_{N_0}^1 \) and all \( i \in N \),

\[ \varphi^2_i(C) = \begin{cases} \frac{1}{|F(l; g(C)) \cup \{l\}) \cap Z(C)|} \sum_{j \in (F(l; g(C)) \cup \{l\})} c_j & \text{if } i \in Z(C) \\ 0 & \text{otherwise.} \end{cases} \]

This rule assigns each leaf an equal portion of the sum of the immediate connection costs of the group, and any other agent zero. It is easy to check that \( \varphi^2 \) satisfies efficiency, non-negativity, independence of following costs, and group independence.
4 Further characterizations

Now we show that how the Bird and the sequential equal contributions rules can be singled out from the family by imposing additional axioms.

4.1 The Bird rule

Our first characterization of the Bird rule is based on first-link consistency.

**Theorem 2.** On $\mathcal{C}^1$, the Bird rule is the only rule satisfying efficiency, group independence, and first-link consistency.

**Proof.** It is clear that the Bird rule satisfies efficiency and group independence. Note that for all $N_0 \in \mathcal{N}_0$, all $C \in \mathcal{C}_{N_0}^1$, and all $i \in N \setminus \{l\}$, $c_i^{-1} = c_i$. Since $\varphi_i^B(C) = c_i = c_i^{-1} = \varphi_i^B(C^{-l})$, the Bird rule satisfies first-link consistency. Conversely, let $\varphi$ be a rule satisfying efficiency, group independence, and first-link consistency. Let $d(i)$ be a function such that if $i \in N \setminus Z(C)$, $d(i) \equiv \max\{h \mid p^h(j) = i \text{ for all } j \in F(i)\}$ and if $i \in Z(C)$, $d(i) \equiv 0$. We show that for all $N_0 \in \mathcal{N}_0$, all $C \in \mathcal{C}_{N_0}^1$, and all $i \in N$, $\varphi_i(C) = c_i$. Two cases are possible.

**Case 1:** $d(i) = 0$. Note that $d(i) = 0$ if and only if $i \in Z(C)$. By group independence and first-link consistency, $\varphi_i(C) = \varphi_i(C^{-P(i)})$. By group independence and efficiency, $\varphi_i(C^{-P(i)}) = c_i$. Therefore, $\varphi_i(C) = c_i$, as desired.

**Case 2:** $d(i) \geq 1$. As induction hypothesis, suppose that for all $j \in N$ such that $d(j) \leq H$, $\varphi_j(C) = c_j$. Now suppose that $d(i) = H + 1$. It is easy to check that

$$\sum_{k \in F(i) \cup \{i\}} \varphi_k(C^{-P(i)}) = \sum_{k \in F(i) \cup \{i\}} c_k. \quad (9)$$

Since for all $k \in F(i)$, $d(k) \leq H$, from the induction hypothesis, $\varphi_k(C) = c_k$. Furthermore, by first-link consistency, for all $k \in F(i)$, $\varphi_k(C^{-P(i)}) = \varphi_k(C)$. Therefore, for all $k \in F(i)$,

$$\varphi_k(C^{-P(i)}) = c_k. \quad (10)$$
Inserting (10) to (9), we have $\varphi_i(C^{-P(i)}) = c_i$. By first-link consistency and group independence, $\varphi_i(C) = \varphi_i(C^{-P(i)})$. Therefore, for all $i \in N$ such that $d(i) = H + 1$, $\varphi_i(C) = c_i$, as desired.

**Remark 2.** First-link consistency requires that a departure of a first-link agent should not affect the cost allocation of his followers. We can formulate a stronger version of the axiom, which requires that the departure should not affect not only his followers but also any other players. With this strengthening of first-link consistency, we can drop group independence from the list appeared in Theorem 2.

Next, we introduce another prevalent property. A rule is a core selection if no coalition of agents can be made better off by building their own network.

**Core selection:** For all $N_0 \in N_0$, all $C \in C_{N_0}$, and all $S \in N$, $\sum_{i \in S} \varphi_i(C) \leq m(C|S_0)$.

We show that the Bird rule is the only core selection of the family.

**Theorem 3.** On $C^1$, the Bird rule is the only core selection from sequential contributions rules.

**Proof.** Bird (1976) shows that the Bird rule is a core selection. Conversely, let $\varphi^\alpha$ be a core selection from sequential contributions rules. For all $N_0 \in N_0$, all $C \in C_{N_0}$, all $i \in N\setminus Z(C)$, and all $\epsilon > 0$, let $\tilde{C}_{e}^{-P(i)} \in C^1_{(N\setminus P(i))0}$ be

$$\tilde{c}_{e}^{-P(i)}_{qw} = \begin{cases} c_w & \text{if } q = i \text{ and } w \in F(i), \\ c_w + \epsilon & \text{if } q \in N\setminus (P(i) \cup \{i\}) \text{ and } w \in F(i), \\ c_{qw} + \epsilon & \text{otherwise.} \end{cases}$$

Note that $\alpha_i(\tilde{C}_{e}^{-P(i)}) = \alpha_i(C)$. For all $j \in F(i)$,

$$\varphi^\alpha_j(\tilde{C}_{e}^{-P(i)}) = c_j + \frac{\epsilon_j - \alpha_i(C)}{|F(i)|}.$$  \hspace{1cm} (11)
Since $\varphi^\alpha$ is a core selection,

$$\varphi_j^\alpha(\bar{C}_{e-P(i)}) \leq c_j + \epsilon. \quad (12)$$

From (11) and (12), we have $\frac{\alpha_i(C) - \alpha_i(C)}{|F(i)|} \leq \epsilon$. Since $\epsilon > 0$ is arbitrary,

$$\alpha_i(C) \geq c_i. \quad (13)$$

By definition of a contributions function,

$$0 \leq \alpha_i(C) \leq c_i. \quad (14)$$

From (13) and (14), for all $i \in N$,

$$\alpha_i(C) = c_i,$$

the desired conclusion.

Next axiom is null responsibility: it requires that if an agent’s immediate connection cost is zero then the cost allocation to the agent is zero.

**Null responsibility**: For all $N_0 \in \mathcal{N}_0$, all $C \in \mathcal{C}_{N_0}$, and all $i \in N$, if $c_i = 0$ then $\varphi_i(C) = 0$.

We show that the Bird rule is the only member of the family satisfying null responsibility. Moreover, the uniqueness holds even if non-negativity is dropped from the list.

**Theorem 4.** On $\mathcal{C}^1$, a rule satisfies efficiency, independence of following costs, group independence, weak first-link consistency, and null responsibility if and only if it is the Bird rule.

**Proof.** It is obvious that the Bird rule satisfies efficiency, independence of following costs, group independence, weak first-link consistency, and null responsibility. Conversely, let $\varphi$ be a rule satisfying the five axioms. For all
We show that for all 
\( i \in N \), let \( D_C(i) \) be a number of arcs in \( g_0(C) \).
We show that for all \( i \in N \), \( \varphi_i(C) = c_i \). Let \( \bar{C} \in C_{N_0}^1 \) be
\[
\bar{c}_{qw} = \begin{cases} 
0 & \text{if } q = i \text{ and } w \in F(i), \\
\infty & \text{if } q \in N \setminus (F(i) \cup \{i\}) \text{ and } w \in F(i), \\
c_{qw} & \text{otherwise}. 
\end{cases}
\]

**Case 1:** \( D_C(i) = 1 \). If \( F(i) = \emptyset \), by efficiency and group independence, \( \varphi_i(C) = c_i \). Now suppose that \( F(i) \neq \emptyset \). By null responsibility, for all \( k \in F(i) \), \( \varphi_k(\bar{C}) = 0 \). By efficiency and group independence, \( \sum_{k \in (F(i) \cup \{i\})} \varphi_k(\bar{C}) = \sum_{k \in (F(i) \cup \{i\})} c_k = c_i \). Therefore, we have \( \varphi_i(\bar{C}) = c_i \). By independence of following costs, \( \varphi_i(C) = \varphi_i(\bar{C}) = c_i \), as desired.

**Case 2:** \( D_C(i) \geq 2 \). As induction hypothesis, suppose that for all \( N_0' \in N_0 \), all \( C' \in C_{N_0}^1 \), and all \( j \in N' \) such that \( D_{C'}(j) \leq H \), \( \varphi_j(C') = c'_j \). Now suppose that \( D_C(i) = H + 1 \). Let \( l = p^H(i) \). By efficiency and group independency,
\[
\sum_{k \in (F(l) \cup \{i\})} \varphi_k(\bar{C}) = \sum_{k \in (F(l) \cup \{i\}) \setminus F(i)} c_k. 
\]
By substituting the conclusion of Case 1 that \( \varphi_l(\bar{C}) = c_l \) into (15), we have
\[
\sum_{k \in F(l)} \varphi_k(\bar{C}) = \sum_{k \in F(l) \setminus F(i)} c_k. 
\]
By weak first-link consistency, there exists a constant \( a \in \mathbb{R} \) such that for all \( k \in F(l) \), \( \varphi_k(\bar{C}) = \varphi_k(\bar{C}^{-l}) + a \). Therefore, (16) can be rewritten as
\[
\sum_{k \in F(l)} \{\varphi_k(\bar{C}^{-l}) + a\} = \sum_{k \in F(l) \setminus F(i)} c_k. 
\]
By efficiency and group independence, \( \sum_{k \in F(l)} \varphi_k(\bar{C}^{-l}) = \sum_{k \in F(l) \setminus F(i)} c_k \). Together with (17), we have \( |F(l)|a = 0 \). Since \( i \in F(l) \), \( |F(l)| \neq 0 \), we have \( a = 0 \). Since \( D_{C^{-l}}(i) = H \), from the induction hypothesis, we have
\[
\varphi_i(\bar{C}^{-l}) = \bar{c}_i^{-l} = c_i. 
\]

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By *weak first-link consistency*, \( \varphi_i(\bar{C}) = \varphi_i(\bar{C}^{-l}) + a = c_i + 0 \) and by *independence of following costs*,

\[
\varphi_i(C) = \varphi_i(\bar{C}) = c_i,
\]

the desired conclusion.

Our next axiom, *independence of irrelevant costs*, requires that a cost allocation to each agent should depend only on his path to the source over a minimal cost spanning tree.

**Independence of irrelevant costs:** For all \( N_0 \in \mathbb{N}_0 \), all \( C, C' \in \mathcal{C}^1_{N_0} \) such that \( g(C) = g(C') \), and all \( i \in N \), if \( c_{qw} = c'_{qw} \) for all \( qw \in g_0(C) \), then \( \varphi_i(C) = \varphi_i(C') \).

Our last characterization of the Bird rule is based on *efficiency*, *null responsibility*, and *independence of irrelevant costs*. Note that *independence of following costs*, *group independence*, and *weak first-link consistency* in Theorem 4 are replaced by *independence of irrelevant costs*.

**Theorem 5.** On \( \mathcal{C}^1 \), the Bird rule is the only rule satisfying *efficiency*, *null responsibility*, and *independence of irrelevant costs*.

**Proof.** It can easily be shown that the Bird rule satisfies *efficiency*, *null responsibility*, and *independence of irrelevant costs*. Conversely, let \( \varphi \) be a rule satisfying *efficiency*, *null responsibility*, and *independence of irrelevant costs*. For all \( N_0 \in \mathbb{N}_0 \), all \( C \in \mathcal{C}^1_{N_0} \), and all \( i \in N \), let \( C^i \in \mathcal{C}^1_{N_0} \) be

\[
c^i_{qw} = \begin{cases} 
c_{qw} & \text{if } (qw) \in g_0(C) \\
0 & \text{if } (qw) \in g(C) \setminus g_0(C) \\
\infty & \text{otherwise.}
\end{cases}
\]

For all \( i \in N \), by *null responsibility*, for all \( k \in N \setminus (P(i) \cup \{i\}) \), we have
\( \varphi_k(C^i) = 0 \). Furthermore, by efficiency, for all \( i \in N \)
\[
\sum_{k \in P(i) \cup \{i\}} c_k = \sum_{k \in P(i) \cup \{i\}} \varphi_k(C^i). \tag{18}
\]
For all \( i \in N \), by independence of irrelevant costs, \( \varphi_i(C^i) = \varphi_i(C) \). Therefore, it suffices to show that for all \( i \in N \), \( \varphi_i(C^i) = c_i \). Two cases are possible.

**Case 1:** \( i \in A(C) \). Since \( P(i) = \emptyset \), we can rewrite (18) as
\[
\varphi_i(C^i) = c_i,
\]
as desired.

**Case 2:** \( i \in N \setminus A(C) \). Since there exists \( p(i) \in N \), from (18), we have
\[
\sum_{k \in P(i)} c_k = \sum_{k \in P(i)} \varphi_k(C^{p(i)}).
\]
By independence of irrelevant costs, for all \( k \in P(i) \), \( \varphi_k(C^{p(i)}) = \varphi_k(C^i) \), which yields,
\[
\sum_{k \in P(i)} c_k = \sum_{k \in P(i)} \varphi_k(C^i). \tag{19}
\]
Inserting (19) to (18),
\[
\sum_{k \in P(i) \cup \{i\}} c_k = \sum_{k \in P(i)} \varphi_k(C^i) + \varphi_i(C^i) = \sum_{k \in P(i)} c_k + \varphi_i(C^i).
\]
Therefore, we have
\[
\varphi_i(C^i) = c_i,
\]
the desired conclusion.

4.2 The sequential equal contributions rule

We present two characterization results of the sequential equal contributions rule based on equal responsibility: it requires that if a connection cost
between two agents on the mst is zero then the cost allocations to the two agents should be the same.

**Equal responsibility:** For all \( N_0 \in N_0 \) and all \( C \in C^1_{N_0} \), if \( i \in N\setminus A(C) \) and \( c_i = 0 \), then \( \varphi_i(C) = \varphi_{p(i)}(C) \), and if \( i \in A(C) \) and \( c_i = 0 \), then \( \varphi_i(C) = 0 \).

We characterize the sequential equal contributions rule from the family by imposing *equal responsibility* additionally. Once again, the uniqueness holds even if *non-negativity* is dropped from the list.

**Theorem 6.** On \( C^1 \), a rule satisfies efficiency, independence of following costs, group independence, weak first-link consistency, and equal responsibility if and only if it is the sequential equal contributions rule.

**Proof.** It is obvious that the sequential equal contributions rule satisfies *efficiency*, *independence of following costs*, *group independence*, *weak first-link consistency*, and *equal responsibility*. Conversely, let \( \varphi \) be a rule satisfying the five axioms. For all \( N_0 \in N_0 \), all \( C \in C^1_{N_0} \), and all \( i \in N \), let \( D_C(i) \) be a number of arcs in \( g_{i0}(C) \). We show that for all \( i \in N \),

\[
\varphi_i(C) = \sum_{h=0}^{D_C(i)-1} \frac{c_{p^h(i)}}{|F(p^h(i))| + 1}.
\]

Let \( \bar{C} \in C^1_{N_0} \) be

\[
\bar{c}_{qw} = \begin{cases} 
0 & \text{if } q = i \text{ and } w \in F(i), \\
\infty & \text{if } q \in N \setminus (F(i) \cup \{i\}) \text{ and } w \in F(i), \\
c_{qw} & \text{otherwise}.
\end{cases}
\]

**Case 1:** \( D_C(i) = 1 \). If \( F(i) = \emptyset \), by *efficiency* and *group independence*, \( \varphi_i(C) = c_i = \frac{c_i}{|F(i)| + 1} \). Now suppose that \( F(i) \neq \emptyset \). By *equal responsibility*, for all \( k \in F(i) \), \( \varphi_k(\bar{C}) = \varphi_i(\bar{C}) \). By *efficiency* and *group independence*,

\[
\sum_{k \in (F(i) \cup \{i\})} \varphi_k(\bar{C}) = \sum_{k \in (F(i) \cup \{i\})} c_k = c_i.
\]

Therefore, we have \( \varphi_i(\bar{C}) = \frac{c_i}{|F(i)| + 1} \). By *independence of following costs*,

\[
\varphi_i(C) = \varphi_i(\bar{C}) = \frac{c_i}{|F(i)| + 1}.
\]
as desired.

**Case 2:** $D_C(i) \geq 2$. As induction hypothesis, suppose that for all $N_0' \in N_0$, all $C' \in C_{N_0'}$, and all $j \in N'$ such that $D_{C'}(j) \leq H$, $\varphi_j(C') = \sum_{h=0}^{D_{C'}(j)-1} \frac{c_{p^h(i)}}{|F(p^h(i))|+1}$. Now suppose that $D_C(i) = H + 1$. Let $l = p^H(i)$. By efficiency and group independency,

$$\sum_{k \in (F(l) \cup \{l\})} \varphi_k(\bar{C}) = \sum_{k \in (F(l) \cup \{l\}) \setminus F(i)} c_k. \tag{20}$$

By substituting the conclusion of Case 1 that $\varphi_l(\bar{C}) = \frac{c_l}{|F(l)|+1}$ into (20), we have

$$\sum_{k \in F(l)} \varphi_k(\bar{C}) = \sum_{k \in F(l) \setminus F(i)} c_k + |F(l)| \frac{c_l}{|F(l)|+1}. \tag{21}$$

By weak first-link consistency, there exists a constant $a \in \mathbb{R}$ such that for all $k \in F(l)$, $\varphi_k(\bar{C}) = \varphi_k(\bar{C}^{-l}) + a$. Therefore, (21) can be rewritten as

$$\sum_{k \in F(l)} \{\varphi_k(\bar{C}^{-l}) + a\} = \sum_{k \in F(l) \setminus F(i)} c_k + |F(l)| \frac{c_l}{|F(l)|+1}. \tag{22}$$

By efficiency and group independence, $\sum_{k \in F(l)} \varphi_k(\bar{C}^{-l}) = \sum_{k \in F(l) \setminus F(i)} c_k$. Together with (22), we have $|F(l)|a = |F(l)| \frac{c_l}{|F(l)|+1}$, or

$$a = \frac{c_l}{|F(l)|+1}. \tag{23}$$

Since $D_{\bar{C}^{-l}}(i) = H$, from the induction hypothesis, we have

$$\varphi_i(\bar{C}^{-l}) = \sum_{h=0}^{H-1} \frac{c_{p^h(i)}}{|F(p^h(i))|+1}. \tag{24}$$

Since $i \in F(l)$, by weak first-link consistency, $\varphi_i(\bar{C}) = \varphi_i(\bar{C}^{-l}) + a$, and combined with (23) and (24), we have

$$\varphi_i(\bar{C}) = \sum_{h=0}^{H-1} \frac{c_{p^h(i)}}{|F(p^h(i))|+1} + \frac{c_l}{|F(l)|+1} = \sum_{h=0}^{H} \frac{c_{p^h(i)}}{|F(p^h(i))|+1}. \tag{25}$$
By independence of following costs,
\[
\varphi_i(C) = \varphi_i(\bar{C}) = \sum_{h=0}^{H} \frac{c_{p^h(i)}}{|F(p^h(i))| + 1},
\]
the desired conclusion. \(\square\)

Our final characterization of the sequential equal contributions rule is based on efficiency, equal responsibility, and independence of irrelevant costs. Once again, independence of following costs, group independence, and weak first-link consistency are replaced by independence of irrelevant costs.

**Theorem 7.** On \(C^1\), the sequential equal contributions rule is the only rule satisfying efficiency, equal responsibility, and independence of irrelevant costs.

**Proof.** It can easily be shown that the sequential equal contributions rule satisfies efficiency, equal responsibility, and independence of irrelevant costs. Conversely, let \(\varphi\) be a rule satisfying the three axioms. For all \(N_0 \in \mathcal{N}_0\), all \(C \in \mathcal{C}^1_{N_0}\), and all \(i \in N\), let \(D_C(i)\) be a number of arcs in \(g_{i0}(C)\). We show that for all \(i \in N\),
\[
\varphi_i(C) = \sum_{h=0}^{D_C(i)-1} \frac{c_{p^h(i)}}{|F(p^h(i))| + 1}.
\]

Let \(C^i \in \mathcal{C}^1_{N_0}\) be
\[
c^i_{qw} = \begin{cases} 
c_{qw} & \text{if } (qw) \in g_{i0}(C) \\
0 & \text{if } (qw) \in g(C) \setminus g_{i0}(C) \\
\infty & \text{otherwise.}
\end{cases}
\]

We divide into two cases.

**Case 1:** \(D_C(i) = 1\). By efficiency, \(\sum_{k \in N} \varphi_k(C^i) = c_i\). Since for all \(k \in F(i)\), \(c_k^i = 0\), by equal responsibility, for all \(k \in F(i)\), \(\varphi_k(C^i) = \varphi_i(C^i)\) and for all
\( k \in N \setminus (F(i) \cup \{i\}), \varphi_k(C^i) = 0 \). Therefore,
\[
c_i = \sum_{k \in F(i) \cup \{i\}} \varphi_k(C^i) + \sum_{k \in N \setminus (F(i) \cup \{i\})} \varphi_k(C^i)
= \sum_{k \in F(i) \cup \{i\}} \varphi_k(C^i)
= \{|F(i)| + 1\} \varphi_i(C^i),
\]
or equivalently,
\[
\varphi_i(C^i) = \frac{c_i}{|F(i)| + 1}. \tag{25}
\]
By independence of irrelevant costs, since \( \varphi_i(C) = \varphi_i(C^i) \), we have from (25),
\[
\varphi_i(C) = \varphi_i(C^i) = \frac{c_i}{|F(i)| + 1},
\]
as desired.

**Case 2:** \( D_C(i) \geq 2 \). As induction hypothesis, suppose that for all \( j \in N \) such that \( D_C(j) \leq H \), \( \varphi_j(C) = \sum_{h=0}^{D_C(j)-1} \frac{c_{p^h(i)}}{|F(p^h(i))|+1} \). Now suppose that \( D_C(i) = H + 1 \). By efficiency,
\[
\sum_{k \in N} \varphi_k(C^{p(i)}) + c_i = \sum_{k \in N} \varphi_k(C^i).
\]
Since for all \( k \in N \setminus F(p^H(i)) \), \( \varphi_k(C^i) = \varphi_k(C^{p(i)}) \), we have
\[
\sum_{k \in F(p^H(i))} \varphi_k(C^{p(i)}) + c_i = \sum_{k \in F(p^H(i))} \varphi_k(C^i).
\]
Furthermore, since for all \( k \in \{p^h(i) \mid h = 1, 2, \ldots, H\} \), \( \varphi_k(C^i) = \varphi_k(C^{p(i)}) \), we have
\[
\sum_{k \in F(i) \cup \{i\}} \varphi_k(C^{p(i)}) + c_i = \sum_{k \in F(i) \cup \{i\}} \varphi_k(C^i). \tag{26}
\]
By equal responsibility, since for all \( k \in F(i) \), \( \varphi_k(C^i) = \varphi_i(C^i) \) and \( \varphi_k(C^{p(i)}) = \varphi_i(C^{p(i)}) \), we have
\[
\{|F(i)| + 1\} \varphi_i(C^{p(i)}) + c_i = \{|F(i)| + 1\} \varphi_i(C^i),
\]

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or equivalently,
\[
\varphi_i(C^i) = \frac{c_i}{|F(i)| + 1} + \varphi_i(C_{p(i)}).
\]

By equal responsibility, \(\varphi_i(C_{p(i)}) = \varphi_{p(i)}(C_{p(i)})\) and by independence of irrelevant costs, \(\varphi_i(C^i) = \varphi_i(C)\) and \(\varphi_{p(i)}(C_{p(i)}) = \varphi_{p(i)}(C)\). Since \(D_C(p(i)) = H\), from the induction hypothesis, \(\varphi_{p(i)}(C) = \sum_{h=0}^{H-1} \frac{c_{p^h(p(i))}}{|F(p^h(p(i)))| + 1}\). Therefore,
\[
\varphi_i(C) = \frac{c_i}{|F(i)| + 1} + \sum_{h=0}^{H-1} \frac{c_{p^h(p(i))}}{|F(p^h(p(i)))| + 1} = \sum_{h=0}^{H} \frac{c_{p^h(i)}}{|F(p^h(i))| + 1},
\]
the desired conclusion.  

\[\square\]

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