The GARCH Option Pricing Model – Theory, Numerical Methods, Evidence and Applications

Jin-Chuan Duan

Risk Management Institute and Department of Finance
National U of Singapore
bizdjc@nus.edu.sg
http://www.rmi.nus.edu.sg/DuanJC

March 2009
The Black-Scholes model

- Asset price process

\[ d \ln(S_t) = \left( r - \delta + \lambda_t \sigma_t - \frac{\sigma_t^2}{2} \right) dt + \sigma_t dW^P_t \]

- Risk-neutral asset price process

\[ d \ln(S_t) = \left( r - \delta - \frac{\sigma_t^2}{2} \right) dt + \sigma_t dW^Q_t \]

- If \( \sigma_t \) is a constant, a European call option with payout \( \max(S_T - K, 0) \) can be priced by

\[
C(S_t; K, T, \sigma) = S_t e^{-\delta (T-t)} N(d_t) - K e^{-r (T-t)} N(d_t - \sigma \sqrt{T-t})
\]

\[
d_t = \frac{\ln(S_t/K) + (r - \delta + \sigma^2/2)(T - t)}{\sigma \sqrt{T-t}}
\]
Implied volatility vs. historical (or realized) volatility

Implied volatility and other volatilities

- The Black-Scholes implied volatility is $\sigma_t^*(K_i, T_j)$ that solves

$$C^{mkt}_t(K_i, T_j) = C(S_t; K_i, T_j, \sigma_t^*(K_i, T_j))$$

- The annualized $\tau$-year realized return variance (quadratic variation) (Divided into $n$ periods with $\Delta t = \tau/n$)

$$\sigma_{t|\tau}^2(n) = \frac{1}{\tau} \sum_{i=1}^{n} \left( \ln \frac{S_{t+i\Delta t}}{S_{t+(i-1)\Delta t}} - \mu_{t+i\Delta t}\Delta t \right)^2$$

$$\approx \frac{1}{\tau} \sum_{i=1}^{n} \left( \ln \frac{S_{t+i\Delta t}}{S_{t+(i-1)\Delta t}} \right)^2$$

Historical volatility is similar but uses returns prior to $t$. 
The annualized $\tau$-year cumulative return variance

\[ \frac{1}{\tau} Var_t \left( \ln \frac{S_{t+\tau}}{S_t} \right) \]

Under the constant-volatility Black-Scholes assumption

\[ \frac{1}{\tau} Var_t \left( \ln \frac{S_{t+\tau}}{S_t} \right) = \sigma^2 = \lim_{n \to \infty} \sigma^2_{t|\tau}(n) = \sigma^*^2(K_i, T_j) \]

Under more general models, $Var_t \left( \ln \frac{S_{t+\tau}}{S_t} \right)$ is not equal to $\sigma^2_{t|\tau}(n)$. Empirically, implied volatilities are different from historical or realized volatility.
Implications of the Black-Scholes theory

Key insight of the general Black-Scholes theory

- The **expected return** (or **risk premium**) of the underlying asset has nothing to do with option values.

- Since the risk premium should be a function of **systematic risk** of the underlying asset, it implies that the composition of the asset risk has no effect on option values.

Implications of the Black-Scholes formula

- The implied volatility of an option is unrelated to its strike price or maturity

- The implied volatility of an option should be the same as the return volatility under the physical law.
Empirical facts on option pricing

1. Implied volatility smile/smirk.

2. Term structure of implied volatilities.

3. Implied volatilities are higher than the physical volatility (historical or realized).

4. The risk-neutral return distribution is more negatively skewed than the physical return distribution.

5. Index options (relative to individual stock options) have more pronounced volatility smile/smirk.
S&P 500 index options’ implied volatilities
S&P100 index option implied vs. historical volatility
Empirical facts on option pricing

Potential causes for the empirical facts

- Non-normal physical return distributions (negatively skewed and leptokurtic) can generate the first two empirical regularities.

- The last three empirical facts, however, point to systematic risk of the underlying asset with the fifth one in particular.

- The existing option pricing models, except for the GARCH option pricing model, are totally silent about the explicit role of the systematic risk (or the asset risk premium) in option pricing.
Systematic risk and option pricing

Define **systematic risk proportion** as the systematic variance over the total variance. Duan and Wei (2007) found

- The level of implied volatility is positively related to the systematic risk proportion of the underlying asset.

- The implied volatility smile/smirk curve becomes more negatively sloped when the systematic risk proportion of the underlying asset is increased.

**Note:** Either findings is at odd with the very core of the Black-Scholes option pricing theory.
The local risk-neutral valuation principle (Duan, 1995)

1. The pricing measure $Q$ and the physical measure $P$ share the null sets.

2. The distributional class of the one-period continuously compounded return (conditional) remains unchanged moving from measure $P$ to $Q$. Moreover, the conditional variance does not change.

3. The expected one-period simple return (conditional and under $Q$) equals the risk-free rate.

Main result: When the one-period return and the stochastic discount factor are jointly lognormal (conditional), then the equilibrium pricing measure satisfies the local risk-neutral valuation principle.
Assume away dividends and $d_{t+1}$ be the one-period stochastic discount factor from $t + 1$ to $t$. Then, the standard pricing result gives rise to

$$S_t = E^P_t (S_{t+1} d_{t+1})$$

Let $m_T = e^{rT} \prod_{i=1}^{T} d_i$. Then $m_t = E^P_t (m_T)$ because $E^P_t (d_{t+1}) = e^{-r}$.

Define a measure $Q$ by $dQ = m_T dP$. Clearly, $Q$ is a probability measure because $m_t$ is positive and $\int 1dQ = 1$.

With respect to $Q$,

$$e^{-r} E^Q_t (S_{t+1}) = e^{-r} E^P_t \left(S_{t+1} \frac{m_{t+1}}{m_t}\right) = E^P_t (S_{t+1} d_{t+1}) = S_t$$
Assume the joint lognormality (conditional) for the asset return and the stochastic discount factor. Then,

\[
\ln \left( \frac{S_{t+1}}{S_t} \right) = \mu_{t+1} + \sigma_{t+1} \epsilon_{t+1}
\]

\[
\ln \left( \frac{m_{t+1}}{m_t} \right) = a_{t+1} + b_{t+1} \epsilon_{t+1} + z_{t+1}
\]

where \( z_{t+1} \) is independent of \( \epsilon_{t+1} \); \( \mu_{t+1}, \sigma_{t+1}, a_{t+1} \) and \( b_{t+1} \) are known at time \( t + 1 \).
Compute the conditional moment generating function of the return innovation under measure $Q$ as follows:

$$E_t^Q \left[ \exp (c \epsilon_{t+1}) \right] = E_t^P \left[ \exp (c \epsilon_{t+1}) \frac{m_{t+1}}{m_t} \right]$$

$$= E_t^P \left[ \exp (a_{t+1} + z_{t+1} + (c + b_{t+1}) \epsilon_{t+1}) \right]$$

$$= q_{t+1} \exp \left( \frac{c^2}{2} + b_{t+1}c \right)$$

where $q_{t+1}$ doesn’t contain $c$.

Setting $c = 0$ gives rise to $q_{t+1} = 1$, and

$$E_t^Q \left[ \exp (c \epsilon_{t+1}) \right] = \exp \left( \frac{c^2}{2} + b_{t+1}c \right)$$

Thus, $\epsilon_{t+1}$ is a $Q$ conditional normal random variable with mean $b_{t+1}$ and variance 1. $b_{t+1}$ can then be determined by applying the risk-neutrality condition.
The NGARCH(1,1) option pricing model

With respect to $P$

$$\ln \left( \frac{S_{t+1}}{S_t} \right) = (r - \delta) + \lambda \sigma_{t+1} - \frac{1}{2} \sigma_{t+1}^2 + \sigma_{t+1} \epsilon_{t+1},$$
$$\sigma_{t+1}^2 = \beta_0 + \beta_1 \sigma_t^2 + \beta_2 \sigma_t^2 (\epsilon_t - \theta)^2,$$
$$\epsilon_{t+1} | \phi_t \overset{P}{\sim} \mathcal{N}(0, 1).$$

With respect to $Q$

$$\ln \left( \frac{S_{t+1}}{S_t} \right) = (r - \delta) - \frac{1}{2} \sigma_{t+1}^2 + \sigma_{t+1} \xi_{t+1},$$
$$\sigma_{t+1}^2 = \beta_0 + \beta_1 \sigma_t^2 + \beta_2 \sigma_t^2 (\xi_t - \lambda - \theta)^2,$$
$$\xi_{t+1} | \phi_t \overset{Q}{\sim} \mathcal{N}(0, 1).$$
Risk-neutral volatility persistence under GARCH

Define $q_{\lambda} = \beta_1 + \beta_2 \left[ 1 + (\theta + \lambda)^2 \right]$.

- The risk-neutral volatility persistence is $q_{\lambda}$ and the physical volatility persistence is $q_0$.

- $q_{\lambda} > q_0$ if $\lambda$ and $\theta$ share the same sign.

- For equity options, the risk-neutral volatility level is expected to be higher than the physical volatility level.

- For equity options, the risk-neutral skewness and kurtosis are expected to be more pronounced than those under the physical law.

- How about currency options?
Implied volatility under GARCH

Parameter values: $\beta_0 = 8 \times 10^{-6}, \beta_1 = 0.85, \beta_2 = 0.08, \theta = 0.5$
Risk-neutral cumulative return volatility under GARCH

Parameter values: $\beta_0 = 8 \times 10^{-6}, \beta_1 = 0.85, \beta_2 = 0.08, \theta = 0.5$
Risk-neutral cumulative return skewness under GARCH

Parameter values: $\beta_0 = 8 \times 10^{-6}, \beta_1 = 0.85, \beta_2 = 0.08, \theta = 0.5$
Risk-neutral cumulative return kurtosis under GARCH

Parameter values: $\beta_0 = 8 \times 10^{-6}, \beta_1 = 0.85, \beta_2 = 0.08, \theta = 0.5$
Implementing the GARCH OPM numerically

- Duan and Simonato (1998) and Duan, Gauthier and Simonato (2001) devised the empirical martingale simulation technique that can be used to price European and path-dependent options.

- Stentoft (2005) developed a primal-simulation method for pricing American options under GARCH.

- Ritchken and Trevor (1999) developed a trinomial-lattice method for pricing European and American style options. Their lattice method has been improved by Cakici and Topyan (2000) and Lyuu and Wu (2005).
• Duan and Simonato (2001) and Duan, Dudley, Gauthier and Simonato (2003) developed the Markov chain techniques for European, American and barrier options.

• Ameur, Breton and Martinez (2006) developed a dynamic programming approach to pricing options under GARCH.

• Heston and Nandi (2000) developed a quasi-analytical formula for European options under a particular GARCH specification.

• Duan, Gauthier and Simonato (1999) and Duan, Gauthier, Sasseville and Simonato (2006) developed analytical approximations for the NGARCH, GJR-GARCH and EGARCH option pricing models.
Empirical evidence supportive of the GARCH option pricing model in a growing literature:

2. Hardle and Hafner (2000, Finance and Stochastics)
9. Duan, Ritchken and Sun (2007, working paper)
Arbitrage-free derivation

- Kallsen and Taqque (1998) constructed a continuous-time GARCH model so that the GARCH option pricing model can be derived via the complete-market argument.

- Duan (2001) devised a semi-recombined binomial-tree GARCH model to obtain the GARCH option pricing model via the complete-market argument.
Conditionally skewed and heavy-tailed innovations

- Duan (1999) used a transformation technique to come up with the GARCH option pricing model that permits conditionally non-Gaussian distributions.

- Duan, Ritchken and Sun (2006, 2007) constructed a GARCH-jump option pricing model based on conditional compound Poisson innovations.

- Christoffersen, Heston and Jacobs (2006) developed a GARCH option pricing model using the Inverse Gaussian innovations.

- Badescu and Kulperger (2006) used both the Esscher transformation and the extended Girsanov principle to deal with general conditional distributions modeled by a nonparametric technique.
Other extensions

- Duan and Pliska (2004) extended the GARCH option pricing model to the co-integrated asset price processes.
- Duan (2005) devised a non-parametric GARCH option pricing model by employing the information-theoretical principle.
- Barone-Adesi, Engle and Mancini (2008) separated the physical and risk-neutral distributions and calibrated them to returns and options.
Static call option delta and vega under GARCH

- Static call option delta is
  \[
  \delta_t = \frac{\partial C(S_t, \sigma_{t+1}^2; K, T)}{\partial S_t} = e^{-r(T-t)} E_t^Q \left( \frac{S_T}{S_t} \mathbb{1}_{\{S_T \geq K\}} \right)
  \]

- Call option vega is
  \[
  \Lambda_t = \frac{\partial C(S_t, \sigma_{t+1}^2; K, T)}{\partial \sigma_{t+1}^2} = \frac{1}{2} e^{-r(T-t)} E_t^Q \left( S_T \mathbb{1}_{\{S_T \geq K\}} \sum_{\tau=t+1}^{T} \left( \frac{\xi_{\tau}}{\sigma_{\tau}} - 1 \right) G_{t+1, \tau-t-1} \right)
  \]

where

\[
G_{t,k} = G_{t,k-1} \left( \beta_1 + \beta_2 (\xi_{\tau} - \theta - \lambda)^2 \right) \quad \text{and} \quad G_0 = 1.
\]
Dynamic call option delta under GARCH

- Dynamic call option delta is

\[ \delta_t^* = \delta_t + 2\Lambda_t \frac{\beta_2(\xi_t - \theta - \lambda)}{S_t} \]

because

\[ \Delta C(S_t, \sigma_{t+1}^2; K, T) \]
\[ \cong \delta_t \Delta S_t + \Lambda_t \Delta \sigma_{t+1}^2 \]
\[ \cong \left( \delta_t + \Lambda_t \frac{\partial \sigma_{t+1}^2}{\partial S_t} \right) \Delta S_t \]
\[ = \left( \delta_t + 2\Lambda_t \frac{\beta_2(\xi_t - \theta - \lambda)}{S_t} \right) \Delta S_t \]
Duan and Yu (1999) applied the GARCH option pricing model to deposit insurance. The model was then used to examine the impact of a bank’s systematic risk on the intrinsic value of a deposit insurance.

Duan and Wei (2005) applied the GARCH option pricing model to executive stock options, which allows them to assess the value of an executive stock option (with or without indexing) in relation to the systematic risk of the firm.
A variance swap at the strike of $K_{t, \text{var}}(\tau)$ is a contract whose payment is linear in the realized variance:

$$[\sigma^2_{t|\tau}(n) - K^2_{t, \text{var}}(\tau)] \times N$$

A volatility swap at the strike of $K_{t, \text{vol}}(\tau)$ is a contract whose payment is linear in the realized volatility:

$$[\sigma_{t|\tau}(n) - K_{t, \text{vol}}(\tau)] \times N$$

Note: $K_{t, \text{var}}(\tau)$ and $K_{t, \text{vol}}(\tau)$ are determined at time $t$. 
Trading variance/volatility swaps

Pricing single-payment variance/volatility swaps

- Pricing requires the use of the risk-neutral pricing measure $Q$ as opposed to the physical probability measure $P$.

- A variance swap at the strike $K_{t,\text{var}}(\tau)$ is fairly priced if

$$K_{t,\text{var}}(\tau) = \sqrt{E_t^Q(\sigma_t^2|\tau(n))}$$

- A volatility swap at the strike $K_{t,\text{vol}}(\tau)$ is fairly priced if

$$K_{t,\text{vol}}(\tau) = E_t^Q(\sigma_t|\tau(n))$$

- By Jensen’s inequality, $K_{t,\text{var}}(\tau) > K_{t,\text{vol}}(\tau)$. 
Performance analysis of shorting variance swaps

Assume VIX is the fair-value strike for the 22-business day variance swaps (to be explained later). The plot suggests shorting variance swaps will be a profitable strategy.
Performance analysis of shorting variance swaps

Short the 22-day variance swap every day from the beginning of 1996 all the way to April 2006.
Replicating the log-price contract by an option portfolio

**Fact 1:**
Consider an option portfolio of European calls and puts weighted inversely proportional to the square of their strike prices. Denote the portfolio value at time \( t \) by

\[
\Pi_t(K_0, t + \tau) = \int_0^{K_0} \frac{P_t(K; t + \tau)}{K^2} dK + \int_{K_0}^{\infty} \frac{C_t(K; t + \tau)}{K^2} dK
\]

where \( 0 < K_0 < \infty \) and \( t + \tau \) is the maturity date. At maturity, it can be shown that

\[
\Pi_{t+\tau}(K_0, t + \tau) = \frac{S_{t+\tau} - K_0}{K_0} - \ln \left( \frac{S_{t+\tau}}{K_0} \right).
\]

At time \( t \), risk-neutral valuation yields

\[
e^{r\tau} \Pi_t(K_0, t + \tau) = \frac{S_t e^{(r-\delta)\tau} - K_0}{K_0} - E^Q_t \left( \ln \frac{S_{t+\tau}}{K_0} \right).
\]
Expected risk-neutral realized return variance

**Fact 2:**
Under diffusion processes, the risk-neutral asset value dynamic is

\[
d \ln S_t = (r - \delta) dt - \frac{1}{2} \sigma_t^2 dt + \sigma_t dW_t^Q
\]

which implies

\[
E_t^Q \left( \ln \frac{S_{t+\tau}}{S_t} \right) = (r - \delta) \tau - \frac{1}{2} E_t^Q \left( \int_t^{t+\tau} \sigma_s^2 ds \right).
\]

Thus,

\[
E_t^Q \left( \sigma_{t+\tau}^2 (\infty) \right) = \frac{1}{\tau} E_t^Q \left( \int_t^{t+\tau} \sigma_s^2 ds \right)
\]

\[
= 2(r - \delta) - \frac{2}{\tau} E_t^Q \left( \ln \frac{S_{t+\tau}}{S_t} \right).
\]
**Fact 2':**

For discrete-time processes with conditional normal innovations, the following result is also true:

\[
E_t^Q \left( \sigma_t^2 \right) = \sum_{i=1}^{n} \frac{\sigma_{t+i\Delta t}^2 \Delta t}{\tau} = 2(r - \delta) - \frac{2}{\tau} E_t^Q \left( \ln \frac{S_{t+\tau}}{S_t} \right)
\]
Link to the fair-value strike of a variance swap

Let $F_t(\tau) = S_t e^{(r-\delta)\tau}$. Facts 1 and 2 (or 2') can be combined to obtain

$$E_t^Q \left( \sigma_{t|\tau}(n) \right) \simeq \frac{2}{\tau} e^{r\tau} \Pi_t(K_0, t + \tau) + \frac{2}{\tau} \left( \ln \frac{F_t(\tau)}{K_0} - \frac{F_t(\tau) - K_0}{K_0} \right)$$

$$\simeq \frac{2}{\tau} e^{r\tau} \Pi_t(K_0, t + \tau) - \frac{1}{\tau} \left( \frac{F_t(\tau)}{K_0} - 1 \right)^2.$$ 

The above result serves as the theoretical basis for the CBOE’s new VIX construction.

1. For general diffusion processes, the first approximate equality is due to using $n < \infty$.

2. The second approximate equality is reasonable when $K_0$ is close to the forward price.
New vs. old VIX

- CBOE launched VIX in 1993 and switched to the new VIX in 2003 (a new methodology applied to the S&500 index options). The old VIX is now under the ticker symbol “VXO”, which continues to use the S&P100 index options. The new VIX tracks VXO reasonably closely, but the new VIX tends to be slightly lower (based on a chart in the CBOE white paper).

- The old VIX uses 8 options to approximate the Black-Scholes implied volatility of a hypothetical at-the-money option with 30 days to maturity. It depends on the Black-Scholes model.
• The new VIX uses all out-of-the-money calls and puts with valid quotes. At-the-money call and put options are also included with their prices averaged. It attempts to gauge the expected risk-neutral realized return variance over next 30 days. The new VIX relies on the concept of static replication, and thus it is NOT subjected to a specific option pricing model.

• The new VIX can in theory be used as the fair-value strike for the 30-day variance swaps, but the old VIX cannot.
Limitations and a way out

- How about the fair-value strike for volatility swaps? No similar result is available. Using the VIX-like volatility directly will in theory overprice volatility swaps.

- Are there enough traded options to reasonably construct the VIX-like volatility? Multiple-payment swaps can present challenges because many long-dated options will be needed.

- A parametric option pricing model is a must for pricing volatility swaps, and in some cases for pricing variance swaps.

- A sensible parametric option pricing model can be useful for devising sophisticated trading strategies.
Expected risk-neutral realized return variance under GARCH

- For $i \geq 1$, $E_t^Q (\sigma^2_{t+i\Delta t}) = \frac{\beta_0 (1-q\lambda^{i-1})}{1-q\lambda} + q\lambda^{i-1} \sigma^2_{t+\Delta t}$

- The fair-value strike of a variance swap under GARCH is

$$\hat{K}_{t,\text{var}}^2 (\tau) = E_t^Q (\sigma^2_{|\tau}(n))$$

$$= \frac{1}{\tau} E_t^Q \left( \sum_{i=1}^{n} \sigma^2_{t+i\Delta t} \Delta t \right)$$

$$= \frac{\beta_0}{1-q\lambda} \left( 1 - \frac{1-q^n\lambda}{n(1-q\lambda)} \right) + \frac{1-q^n\lambda}{n(1-q\lambda)} \sigma^2_{t+\Delta t}$$
Fitting the S&P500 return and VIX series ($\tau = 1/12$ or 22 business days)

Parameter estimates (1/15/1996 – 12/31/2002): $\beta_0 = 3.5539 \times 10^{-6}$, $\beta_1 = 0.914988803$, $\beta_2 = 0.0400086565$, $\theta = 0.7574614953$, $\lambda = 0.2237641279$ and $\sigma_{1/15/96} = 0.0697282369$
Predicting the profit from shorting using GARCH

The regression function:
\[ VIX_t^2 - \sigma_t^2_{(1/12)}(22) = 189.168489 + 0.660731525 \times (VIX_t^2 - \hat{K}_{t,\text{var}}^2(1/12)) \]
Out-sample relationship
The GARCH-based shorting uses the magnitude of the predicted shorting profit to move up or down the position size. The positions have been adjusted so that the average position size is one variance swap.


CBOE white paper on VIX, CBOE website.


Duan, J., P. Ritchken and Z. Sun, 2007, “Jump Starting GARCH: Pricing and Hedging Options with Jumps in Returns and Volatilities,” University of Toronto and Case Western Reserve University working paper.


